Semantics for Second-Order Modal Object Theory

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Semantics

Interpretations

Metalanguage and Metatheory: first-order set theory with urelements and a Hilbert epsilon operator $\bar{\varepsilon}$. The $\bar{\varepsilon}$-operator is to be understood classically: it is a variable-binding operator that can bind a variable in a semantic formula, where the resulting metalinguistic term acts as a choice function which picks out an arbitrary member of the domain (over which the variable ranges) that satisfies the semantic formula, if there is one.[1] We don't require that the $\bar{\varepsilon}$-operator be extensional, i.e., we don't require that $\forall \boldsymbol{r}(\Phi \equiv \boldsymbol{B}) \rightarrow \bar{\varepsilon}\boldsymbol{r}\Phi = \bar{\varepsilon}\boldsymbol{r}\boldsymbol{B}$.[2]

Interpretations: With such a metalanguage, we may specify a formal semantic interpretation $\mathcal{I}$ of the object language as a structure of the following form:

$$\mathcal{I} = \langle \mathbf{D}, \mathbf{R}, \mathbf{W}, \boldsymbol{T}, \boldsymbol{F}, \mathbf{ext}_{\boldsymbol{w}}, \mathbf{enc}_{\boldsymbol{w}}, \mathbf{ex}_{\boldsymbol{w}}, \mathbf{V}, \mathbf{C} \rangle,$$

whose elements are as follows:

1. $\mathbf{D}$ is a nonempty domain of primitive individuals. In what follows, we use $\boldsymbol{o}_1, \boldsymbol{o}_2, \ldots$ to range over the elements of $\mathbf{D}$.

2. $\mathbf{R}$ is the general union of nonempty domains $\mathbf{R}_n$ of primitive n-ary relations, i.e., $\mathbf{R} = \bigcup_{n \geq 0} \mathbf{R}_n$. In what follows, we use $\boldsymbol{r}^{\boldsymbol{n}}$ to range over the elements of $\mathbf{R}_n$ ($n \geq 1$) and $\boldsymbol{p}$ to range over the elements of $\mathbf{R}_0$.

3. $\mathbf{W}$ is a nonempty domain of possible worlds and contains a distinguished element $\boldsymbol{w}_0$, known as the actual world. In what follows, we use $\boldsymbol{w}$ to range over the elements of $\mathbf{W}$.

4. $\boldsymbol{T}$ is the truth-value The True.

5. $\boldsymbol{F}$ is the truth-value The False.

6. $\mathbf{ext}_{\boldsymbol{w}}$ is a binary function, indexed to its second argument, that assigns each n-ary relation $\boldsymbol{r}^n$ in $\mathbf{R}_n$ ($n \geq 1$) an *exemplification extension* at each possible world $\boldsymbol{w}$, as follows: $\mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{r}^n)$ is a set of n-tuples of the form $\langle \boldsymbol{o}_1, \ldots, \boldsymbol{o}_n \rangle$, i.e., $\mathbf{ext}_{\boldsymbol{w}} : \mathbf{R}_n \times \mathbf{W} \rightarrow \wp(\mathbf{D}^n)$. By convention, when $n = 1$, $\mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{r}^1)$ is a subset of $\mathbf{D}$.

7. $\mathbf{enc}_{\boldsymbol{w}}$ is a binary function, indexed to its second argument, that assigns each n-ary relation $\boldsymbol{r}^n$ in $\mathbf{R}$ ($n \geq 1$) an *encoding extension* at each possible world $\boldsymbol{w}$, as follows: $\mathbf{enc}_{\boldsymbol{w}}(\boldsymbol{r}^n)$ is a set of n-tuples of the form $\langle \boldsymbol{o}_1, \ldots, \boldsymbol{o}_n \rangle$, i.e., $\mathbf{enc}_{\boldsymbol{w}} : \mathbf{R}_n \times \mathbf{W} \rightarrow \wp(\mathbf{D}^n)$. Again, by convention, when $n = 1$, $\mathbf{enc}_{\boldsymbol{w}}(\boldsymbol{r}^1)$ is a subset of $\mathbf{D}$.

8. $\mathbf{ex}_{\boldsymbol{w}}$ is a binary function, indexed to its second argument, that assigns each 0-ary relation $\boldsymbol{p}$ in $\mathbf{R}_0$ an *extension* at each possible world $\boldsymbol{w}$, as follows: $\mathbf{ex}_{\boldsymbol{w}}(\boldsymbol{p})$ is either $\boldsymbol{T}$ or $\boldsymbol{F}$; i.e., $\mathbf{ex}_{\boldsymbol{w}} : \mathbf{R}_0 \times \mathbf{W} \rightarrow \{\boldsymbol{T}, \boldsymbol{F}\}$.

[1] So, for example, if Φ is a formula of the metalanguage that places a condition on the semantic variable $\boldsymbol{r}$, then $\bar{\varepsilon}\boldsymbol{r}\Phi$ is a term that denotes an entity in the domain over which $\boldsymbol{r}$ ranges that satisfies Φ, if there is one. Terms of this kind will be used in D4 and D5 below, in Section . We require only the usual axiom for the $\bar{\varepsilon}$-calculus, namely, $\exists \boldsymbol{r}\Phi \rightarrow \Phi_{\boldsymbol{r}}^{\bar{\varepsilon}\boldsymbol{r}\Phi}$. This asserts that if there exists an $\boldsymbol{r}$ such that Φ, then an $\boldsymbol{r}$ such that Φ is such that Φ.

[2] We do minimally require that:

- $\exists \boldsymbol{r}\Phi \rightarrow \exists \boldsymbol{r}(\boldsymbol{r} = \bar{\varepsilon}\boldsymbol{r}\Phi)$
- $\boldsymbol{r} = \bar{\varepsilon}\boldsymbol{r}\Phi \rightarrow \Phi$

These assert, respectively, that if there exists an $\boldsymbol{r}$ such that Φ, then there exists something that is (identical to) an $\boldsymbol{r}$ such that Φ, and that if an entity $\boldsymbol{r}$ is (identical to) an $\boldsymbol{r}$ such that Φ, then Φ.

9. **V** is an interpretation function that assigns a meaning to the primitive terms of our language:

   - where τ is any individual constant, $\mathbf{V}(\tau)$ is an element of **D**, and
   - where τ is any n-ary relation constant ($n \geq 0$), then $\mathbf{V}(\tau)$ is an element of $\mathbf{R}_n$, and so an element of **R**.

10. **C** is a choice function that takes, as argument, any semantic formula Φ having a single free variable that ranges over some domain $\mathbf{R}_n$ ($n \geq 0$), and returns an arbitrary but determinate value in $\mathbf{R}_n$ that satisfies Φ if there is one, and is undefined otherwise. For example, if $n \neq 0$, then our metalinguistic notation $\bar{\varepsilon}\boldsymbol{r}^n\Phi$ denotes $\mathbf{C}(\Phi)$, where the latter is an arbitrary but determinate relation in $\mathbf{R}_n$ that satisfies Φ, if there is one. And if $n = 0$, then $\bar{\varepsilon}\boldsymbol{p}\Phi$ denotes $\mathbf{C}(\Phi)$, where the latter is an arbitrary but determinate proposition in $\mathbf{R}_0$ that satisfies Φ, if there is one.

Henceforth, when we refer to any the above elements, we assume that they are relative to some salient interpretation. Thus, for example, when we refer **V**, we assume that **V** is the interpretation function of some salient or fixed interpretation $\mathcal{I}$.

Assignments to Variables

Given any interpretation $\mathcal{I}$, we let an *assignment* function to the variables be a function $f_{\mathcal{I}}$ that maps each individual variable to an element of **D** and maps each n-ary relation variable ($n \geq 0$) to an element of $\mathbf{R}_n$. Henceforth, we shall suppress the subscript on $f_{\mathcal{I}}$, though the reader should remember that all such assignment functions are defined relative to a given interpretation.

Let f be an assignment to the variables, and consider any variable α. Since α might be an individual variable or a relation variable, let $\boldsymbol{e}$ be a variable ranging over the entities in $\mathbf{D} \cup \mathbf{R}$ with the understanding that $\boldsymbol{e}$ is some entity in the domain over which α ranges. Then we may define the variable assignment just like f except that it assigns to the variable α the entity $\boldsymbol{e}$, written $f[\alpha/\boldsymbol{e}]$, in one two ways. If an assignment function f is represented as a set of ordered pairs, then where α is a variable and $\boldsymbol{e}$ is an entity from the domain over which α ranges:

$$f[\alpha/\boldsymbol{e}] = (f \sim \langle \alpha, f(\alpha) \rangle) \cup \{\langle \alpha, \boldsymbol{e} \rangle\}$$

I.e., $f[\alpha/\boldsymbol{e}]$ is the result of removing the pair $\langle \alpha, f(\alpha) \rangle$ from f and replacing it with the pair $\langle \alpha, \boldsymbol{e} \rangle$. Alternatively, we can define $f[\alpha/\boldsymbol{e}]$ functionally, where β is a variable ranging over the same domain as α, as:

$$f[\alpha/e](\beta) = \begin{cases} f(\beta), \text{if} \beta \neq \alpha \\ \boldsymbol{e}, \quad \text{if} \beta = \alpha \end{cases}$$

Since we have two kinds of variables, we shall see this definition used in two contexts.

> Context 1: α is an individual variable, e.g., x, and the domain over which α ranges is **D**. If we are discussing the assignment-relative truth conditions of a formula in the object-language in which the variable x appears, we use $f[x/\boldsymbol{o}]$ to refer to the assignment just like f except that it assigns to x the individual $\boldsymbol{o}$; if we are using the metavariable ν, which ranges over individual variables, to discuss the assignment-relative truth conditions of a formula schema specified in terms of ν, we use $f[\nu/\boldsymbol{o}]$ to refer to the assignment just like f except that it assigns to ν the object $\boldsymbol{o}$.

> Context 2: α is an n-ary relation variable, e.g., F^n, and the domain over which α ranges is $\mathbf{R}_n$. If we are discussing, relative to an assignment f, the truth conditions of a formula in the object-language involving the variable F^n, we use $f[F^n/\boldsymbol{r}^n]$ to refer to the assignment just like f except that it assigns to F^n the n-ary relation $\boldsymbol{r}^n$ in $\mathbf{R}_n$.

Moreover, we extend this definition in the usual way so that $f[\nu_i/\boldsymbol{o}_i]_{i=1}^n$ is the variable assignment just like f but which assigns the objects $\boldsymbol{o}_1, \ldots, \boldsymbol{o}_n$, respectively, to the variables $\nu_1, \ldots, \nu_n$, for $1 \leq i \leq n$. We leave this definition as an exercise.

Denotation, and Truth, with respect to $\mathcal{I}$ and f

Given an interpretation $\mathcal{I}$ and an assignment f, we assign denotations to the terms and world-relative truth conditions to the formulas by defining the following notions simultaneously:

$\boldsymbol{d}_{\mathcal{I},f}(\tau)$, i.e., the denotation of τ relative to $\mathcal{I}$ and f

$\boldsymbol{w} \models_{\mathcal{I},f} \varphi$, i.e., *$\varphi$ is true at $\boldsymbol{w}$ under $\mathcal{I}$* and f

We give the base clauses first and then the recursive clauses (and in what follows, we adopt the convention of omitting the arity superscript on a relation symbol on all occurrences *after* its first use in a semantic formula, whenever this can be done without ambiguity):

Base Clauses

D1. If τ is a constant, then $\boldsymbol{d}_{\mathcal{I},f}(\tau) = \mathbf{V}(\tau)$

D2. If τ is a variable, then $\boldsymbol{d}_{\mathcal{I},f}(\tau) = f(\tau)$

T1. If φ is a formula of the form $\Pi\kappa_1 \ldots \kappa_n$ $(n \geq 1)$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\exists \boldsymbol{r}^n \exists \boldsymbol{o}_1 \ldots \exists \boldsymbol{o}_n(\boldsymbol{r} = \boldsymbol{d}_{\mathcal{I},f}(\Pi)\ \&\ \boldsymbol{o}_1 = \boldsymbol{d}_{\mathcal{I},f}(\kappa_1)\ \&\ \ldots\ \&\ \boldsymbol{o}_n = \boldsymbol{d}_{\mathcal{I},f}(\kappa_n)\ \&\ \langle \boldsymbol{o}_1, \ldots, \boldsymbol{o}_n \rangle \in \mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{r}))$

T2. If φ is a formula of the form $\kappa_1 \ldots \kappa_n \Pi^n$ $(n \geq 1)$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\exists \boldsymbol{o}_1 \ldots \exists \boldsymbol{o}_n \exists \boldsymbol{r}^n(\boldsymbol{o}_1 = \boldsymbol{d}_{\mathcal{I},f}(\kappa_1)\ \&\ \ldots\ \&\ \boldsymbol{o}_n = \boldsymbol{d}_{\mathcal{I},f}(\kappa_n)\ \&\ \boldsymbol{r} = \boldsymbol{d}_{\mathcal{I},f}(\Pi)\ \&\ \langle \boldsymbol{o}_1, \ldots, \boldsymbol{o}_n \rangle \in \mathbf{enc}_{\boldsymbol{w}}(\boldsymbol{r}))$

T3. If φ is a 0-ary relation constant or 0-ary relation variable Π, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\mathbf{ex}_{\boldsymbol{w}}(\boldsymbol{d}_{\mathcal{I},f}(\Pi)) = \boldsymbol{T})$

Recursive Clauses

T4. If φ is a formula of the form $[\lambda\, \psi]$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\boldsymbol{w} \models_{\mathcal{I},f} \psi$

T5. If φ is a formula of the form $\neg\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if it is not the case that $\boldsymbol{w} \models_{\mathcal{I},f} \psi$, i.e., iff $\boldsymbol{w} \not\models_{\mathcal{I},f} \psi$

T6. If φ is a formula of the form $\psi \to \chi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if either it is not the case that $\boldsymbol{w} \models_{\mathcal{I},f} \psi$ or it is the case that $\boldsymbol{w} \models_{\mathcal{I},f} \chi$, i.e., iff either $\boldsymbol{w} \not\models_{\mathcal{I},f} \psi$ or $\boldsymbol{w} \models_{\mathcal{I},f} \chi$

T7. If φ is a formula of the form $\forall\alpha\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\forall \boldsymbol{e}(\boldsymbol{w} \models_{\mathcal{I},f[\alpha/\boldsymbol{e}]} \psi)$

T8. If φ is a formula of the form $\Box\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\forall \boldsymbol{w}'(\boldsymbol{w}' \models_{\mathcal{I},f} \psi)$

T9. If φ is a formula of the form $\mathcal{A}\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\boldsymbol{w}_0 \models_{\mathcal{I},f} \psi$

D3. If τ is a description of the form $\imath\nu\varphi$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \begin{cases} \boldsymbol{o}, \text{ if } \boldsymbol{w}_0 \models_{\mathcal{I},f[\nu/\boldsymbol{o}]} \varphi\ \&\ \forall \boldsymbol{o}'(\boldsymbol{w}_0 \models_{\mathcal{I},f[\nu/\boldsymbol{o}']} \varphi \to \boldsymbol{o}' = \boldsymbol{o}) \\ \text{undefined, otherwise} \end{cases}$$

where $\boldsymbol{o}'$ also ranges over the entities in $\mathbf{D}$

D4. If τ is an n-ary λ-expression $(n \geq 1)$ of the form $[\lambda \nu_1 \ldots \nu_n\, \varphi]$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \begin{cases} \bar{\varepsilon}\boldsymbol{r}^n \forall \boldsymbol{w} \forall \boldsymbol{o}_1 \ldots \forall \boldsymbol{o}_n(\langle \boldsymbol{o}_1, \ldots, \boldsymbol{o}_n \rangle \in \mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{r}) \equiv \boldsymbol{w} \models_{\mathcal{I},f[\nu_i/\boldsymbol{o}_i]_{i=1}^n} \varphi), \\ \quad \text{if there is one} \\ \text{undefined, otherwise} \end{cases}$$

where $\bar{\varepsilon}\boldsymbol{r}\Phi = \mathbf{C}(\Phi)$ and $\mathbf{C}$ is the choice function of the interpretation.

D5. If τ is a 0-ary λ-expression of the form $[\lambda\, \varphi]$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \bar{\varepsilon}\boldsymbol{p}\forall \boldsymbol{w}(\mathbf{ex}_{\boldsymbol{w}}(\boldsymbol{p}) = \boldsymbol{T} \;\equiv\; \boldsymbol{w} \models_{\mathcal{I},f} \varphi)$$

where $\bar{\varepsilon}\boldsymbol{p}\Phi = \mathbf{C}(\Phi)$ and $\mathbf{C}$ is the choice function of the interpretation.

D6. If τ is 0-ary relation term Π^0, i.e., if τ is a formula φ, then:

- if φ is a 0-ary relation constant or a 0-ary relation variable, then $\boldsymbol{d}_{\mathcal{I},f}(\tau)$ is given by D1 – D2
- if φ is a formula of the form $[\lambda\, \varphi]$, then $\boldsymbol{d}_{\mathcal{I},f}(\tau)$ is given by D5
- if φ is a formula of any other form, then $\boldsymbol{d}_{\mathcal{I},f}(\tau) = \boldsymbol{d}_{\mathcal{I},f}([\lambda\, \varphi])$

We now give an informal explanation of the elements of this definition.

Fix an interpretation $\mathcal{I}$ and an assignment f. Then let us say that an entity $\boldsymbol{e}$ in the domain over which a variable α ranges *satisfies*$_{\mathcal{I},f}$ φ *at* world $\boldsymbol{w}$ just in case $\boldsymbol{w} \models_{\mathcal{I},f[\alpha/\boldsymbol{e}]} \varphi$. And let us say that the n-tuple of individuals $\langle \boldsymbol{o}_1,\ldots,\boldsymbol{o}_n\rangle$ *satisfies*$_{\mathcal{I},f}$ φ *at* world $\boldsymbol{w}$ just in case $\boldsymbol{w} \models_{\mathcal{I},f[\alpha_i/\boldsymbol{e}_i]_{i=1}^{n}} \varphi$. Then we may give more intuitive readings of D1–D6 and T1–T9 as follows:

D1. The denotation of a primitive constant is what $\mathbf{V}$ assigns that constant.

D2. The denotation of a variable is what f assigns that variable.

T1. An exemplification formula is true at world $\boldsymbol{w}$ iff there are denotations for all the terms of the formula and the n-tuple of the individuals denoted by the individual terms is an element of the exemplification extension at $\boldsymbol{w}$ of the relation denoted by the relation term.

T2. An encoding formula is true at world $\boldsymbol{w}$ iff there are denotations for all the terms of the formula the n-tuple of the individuals denoted by the individual terms is an element of the encoding extension at $\boldsymbol{w}$ of the relation denoted by the relation term.

T3. A formula consisting solely of a 0-ary relation constant or variable is true at $\boldsymbol{w}$ iff the exemplification extension at $\boldsymbol{w}$ of the proposition denoted by the formula is the truth value $\boldsymbol{T}$.

T4. The truth conditions of *that*-φ at $\boldsymbol{w}$ are the same as the truth conditions of φ at $\boldsymbol{w}$.

T5. A negated formula is true at $\boldsymbol{w}$ iff the unnegated formula fails to be true at $\boldsymbol{w}$.

T6. A conditional formula is true at $\boldsymbol{w}$ iff either the antecedent fails to be true at $\boldsymbol{w}$ or the consequent is true at $\boldsymbol{w}$.

T7. A universally quantified formula $\forall\alpha\psi$ is true at $\boldsymbol{w}$ iff every element of the domain over which α ranges satisfies ψ at $\boldsymbol{w}$.

T8. A modal formula $\Box\psi$ is true at $\boldsymbol{w}$ iff ψ is true at every possible world $\boldsymbol{w}'$.

T9. A formula of the form $\mathcal{A}\psi$ is true at $\boldsymbol{w}$ iff ψ is true at the actual world $\boldsymbol{w}_0$.

D3. A description $\imath\nu\varphi$ denotes an individual $\boldsymbol{o}$ just in case $\boldsymbol{o}$ uniquely satisfies ψ at the actual world $\boldsymbol{w}_0$.

D4. A n-ary λ-expression $[\lambda\nu_1 \ldots \nu_n\, \varphi]$ $(n \geq 1)$ denotes a relation $\boldsymbol{r}$ whose exemplification extension at any world $\boldsymbol{w}$ consists of all and only those n-tuples that satisfy φ at $\boldsymbol{w}$

D5. A 0-ary λ-expression $[\lambda\, \varphi]$ denotes a proposition whose exemplification extension at any world $\boldsymbol{w}$ is The True just in case φ is true at $\boldsymbol{w}$.

D6. Since a 0-ary term is a formula φ, then:

- if φ is a 0-ary constant or a variable, then its denotation is given in D1 and D2
- if φ is a 0-ary λ-expression, then its denotation is given in D5
- if φ is a formula having any other form, then it denotes what $[\lambda\, \varphi]$ denotes.

Given the usual definitions of conjunction (&), disjunction ($\vee$), the biconditional ($\equiv$), the existential quantifier ($\exists\alpha$), and the possibility operator ($\Diamond$), the above definition yields the following facts:

T10. if φ is a formula of the form $\psi \,\&\, \chi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only both $\boldsymbol{w} \models_{\mathcal{I},f} \psi$ and $\boldsymbol{w} \models_{\mathcal{I},f} \chi$

T11. if φ is a formula of the form $\psi \vee \chi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only either $\boldsymbol{w} \models_{\mathcal{I},f} \psi$ or $\boldsymbol{w} \models_{\mathcal{I},f} \chi$

T12. if φ is a formula of the form $\psi \equiv \chi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only ($\boldsymbol{w} \models_{\mathcal{I},f} \psi$ if and only if $\boldsymbol{w} \models_{\mathcal{I},f} \chi$).

T13. If φ is a formula of the form $\exists\alpha\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if
$\exists \boldsymbol{e}(\boldsymbol{w} \models_{\mathcal{I},f[\alpha/\boldsymbol{e}]} \psi)$

T14. If φ is a formula of the form $\Diamond\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if
$\exists \boldsymbol{w}'(\boldsymbol{w}' \models_{\mathcal{I},f} \psi)$

Truth, Validity, and Logical Consequence

We now define φ is *true under $\mathcal{I}$ and f* (written $\models_{\mathcal{I},f} \varphi$) if and only if φ is true at the distinguished actual world $\boldsymbol{w}_0$ under $\mathcal{I}$ and f. Formally:

Definition of Truth under $\mathcal{I}$ and f
$\models_{\mathcal{I},f} \varphi \ =_{df} \ \boldsymbol{w}_0 \models_{\mathcal{I},f} \varphi$

Following the usual practice, we define φ is *true under $\mathcal{I}$* (written $\models_{\mathcal{I}} \varphi$) if and only if for every f, φ is true under $\mathcal{I}$ and f. Formally:

Definition of Truth Under $\mathcal{I}$
$\models_{\mathcal{I}} \varphi \ =_{df} \models_{\mathcal{I},f} \varphi$, for every f

Clearly, when $\models_{\mathcal{I}} \varphi$, it follows that for every f, $\boldsymbol{w}_0 \models_{\mathcal{I},f} \varphi$, by definition of $\models_{\mathcal{I},f} \varphi$. Moreover, if φ is not true under $\mathcal{I}$, then some assignment f is such that $\boldsymbol{w}_0 \not\models_{\mathcal{I},f} \varphi$ and we write $\not\models_{\mathcal{I}} \varphi$. We may also say:

> A formula φ is *false under $\mathcal{I}$* iff no assignment function f is such that $\models_{\mathcal{I},f} \varphi$, i.e., iff no assignment function f is such that $\boldsymbol{w}_0 \models_{\mathcal{I},f} \varphi$

Next, we define φ is *valid* or *logically true* (written $\models \varphi$) if and only if φ is true under every interpretation $\mathcal{I}$. Formally:

Definition of Validity (= Logical Truth)
$\models \varphi \ =_{df} \models_{\mathcal{I}} \varphi$, for every $\mathcal{I}$

Clearly, given our previous definitions, it follows that:

$\models \varphi$ if and only if for every $\mathcal{I}$ and f, $\models_{\mathcal{I},f} \varphi$, i.e.,

$\models \varphi$ if and only if for every $\mathcal{I}$ and f, $\boldsymbol{w}_0 \models_{\mathcal{I},f} \varphi$

In what follows, when we say that a schema is valid, we mean that all of its instances are valid. Clearly, if a formula φ is not valid, then for some interpretation $\mathcal{I}$ and assignment f, $\boldsymbol{w}_0 \not\models_{\mathcal{I},f} \varphi$.

Finally, we complete our series of semantics definitions with the following:

- φ is *satisfiable* if and only if there is some interpretation $\mathcal{I}$ and assignment f such that φ is true$_{\mathcal{I},f}$, i.e., iff $\exists\mathcal{I}\,\exists f(\models_{\mathcal{I},f} \varphi)$.
- φ *logically implies* ψ (or ψ is a *logical consequence* of φ) just in case, for every interpretation $\mathcal{I}$ and assignment f, if φ is true$_{\mathcal{I},f}$, then ψ is true$_{\mathcal{I},f}$. Formally, $\varphi \models \psi \ =_{df} \ \forall\mathcal{I}\forall f(\models_{\mathcal{I},f} \varphi \rightarrow \ \models_{\mathcal{I},f} \psi)$
- φ and ψ are *logically equivalent* just in case both $\varphi \models \psi$ and $\psi \models \varphi$.
- φ is a *logical consequence* of a set of formulas Γ just in case, for every interpretation $\mathcal{I}$ and assignment f, if every member of Γ is true$_{\mathcal{I},f}$, then φ is true$_{\mathcal{I},f}$. Formally, $\Gamma \models \varphi \ =_{df} \ \forall\mathcal{I}\forall f[\forall\psi(\psi \in \Gamma \rightarrow \ \models_{\mathcal{I},f} \psi) \rightarrow \ \models_{\mathcal{I},f} \varphi]$

By convention, when $\psi_1, \ldots, \psi_n$ ($n \geq 0$) are the members of Γ, we shall write $\psi_1, \ldots, \psi_n \models \varphi$ instead of $\{\psi_1, \ldots, \psi_n\} \models \varphi$. Clearly, then, when Γ is empty, the definition of logical consequence reduces to that of logical validity: when φ is a logical consequence of the empty set of formulas, it is valid. Note also that we shall write $\Gamma, \varphi \models \psi$ to indicate that $\Gamma \cup \{\varphi\} \models \psi$.

The above definitions embody the traditional semantic conception of truth derived from Tarski 1933 and 1944, though we've (a) extended them to apply to our modal language, and (b) utilized domains of interpretation that contain primitive hyperintensional entities (i.e., *n*-ary relations).

Simple Example

We begin with an example that shows how the definition of $\models_{\mathcal{I},f} \varphi$ works. Let φ be the formula $\Box(\neg p \rightarrow Qb)$ and fix an interpretation $\mathcal{I}$ and pick some assignment f to the variables. Then, applying the above definitions, we have:

$\models_{\mathcal{I},f} \Box(\neg p \rightarrow Qb)$

iff $\boldsymbol{w}_0 \models_{\mathcal{I},f} \Box(\neg p \rightarrow Qb)$	(By the definition of $\models_{\mathcal{I},f} \varphi$)
iff $\forall \boldsymbol{w}(\boldsymbol{w} \models_{\mathcal{I},f} \neg p \rightarrow Qb)$	(By T8)
iff $\forall \boldsymbol{w}(\boldsymbol{w} \not\models_{\mathcal{I},f} \neg p \ \vee\ \boldsymbol{w} \models_{\mathcal{I},f} Qb)$	(By T6)
iff $\forall \boldsymbol{w}(\neg \boldsymbol{w} \not\models_{\mathcal{I},f} p \ \vee\ \boldsymbol{w} \models_{\mathcal{I},f} Qb)$	(By T5)
iff $\forall \boldsymbol{w}(\boldsymbol{w} \models_{\mathcal{I},f} p \ \vee\ \boldsymbol{w} \models_{\mathcal{I},f} Qb)$	(Eliminate double negation)
iff $\forall \boldsymbol{w}(\mathbf{ex}_{\boldsymbol{w}}(\boldsymbol{d}_{\mathcal{I},f}(p)) = \boldsymbol{T} \ \vee\ \boldsymbol{w} \models_{\mathcal{I},f} Qb)$	(By T3)
iff $\forall \boldsymbol{w}(\mathbf{ex}_{\boldsymbol{w}}(\boldsymbol{d}_{\mathcal{I},f}(p)) = \boldsymbol{T} \ \vee\ \boldsymbol{d}_{\mathcal{I},f}(b) \in \mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{d}_{\mathcal{I},f}(Q)))$	(By T1)

In other words, $\Box(p \rightarrow Qb)$ is true under I and f just in case for every possible world $\boldsymbol{w}$, either the exemplification extension at $\boldsymbol{w}$ of the proposition denoted$_{\mathcal{I},f}$ by p is the truth value $\boldsymbol{T}$ or the object denoted$_{\mathcal{I},f}$ by b is in the exemplification extension at $\boldsymbol{w}$ of the property denoted$_{\mathcal{I},f}$ by Q. As an exercise, the reader is encouraged to state the truth conditions of a complex formula that has an encoding formula as a subformula, to illustrate how clause T2 (in the definition of $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$) works.